Celestial mechanics and dynamics

Solar System and Planetary Astronomy
Astro 9601
Topics to be covered

- The two-body problem (2.1)
- The three-body problem (2.2)
- Perturbations and resonances (2.3)
- Long-term stability and chaos (2.4)
- Tides (2.6) briefly: to be revisited in Ch 6
- Dissipative forces and small bodies (2.7)
The Two-Body problem

Consider two massive bodies moving only under their own mutual gravity.

Body 1 has mass $m_1$ and has an initial vector position $r_1$ and velocity $v_1$ relative to the (arbitrary) origin.

Body 2 has mass $m_2$ and has an initial vector position $r_2$ and velocity $v_2$ relative to the (arbitrary) origin.

What is their subsequent motion?
The Two-Body problem

Newton’s second law: \[ F = ma \]

Newton’s third law: \[ \vec{F}_{12} = -\vec{F}_{21} \]

Newton’s law of gravity:

\[ \vec{F} = -\nabla \frac{Gm_1 m_2}{|\vec{r}_2 - \vec{r}_1|} = -\nabla U \]

\[ |\vec{F}_{12}| = |\vec{F}_{21}| = \frac{Gm_1 m_2}{|\vec{r}_2 - \vec{r}_1|^2} \]
The radial equation

If one body is much more massive than the other, the solution is a conic section with the central body (Sun) at one focus. The precise nature is determined by a parameter called the eccentricity \( e \).

If \( e=0 \), it is a circle, if \( 0<e<1 \), it is an ellipse, if \( e=1 \), it is a parabola, if \( e>1 \) it is a hyperbola.
Inclination

• The orbit is confined to a 2D plane about the Sun. The angle between the orbit and a reference plane (usually the Earth’s orbit) is called the *inclination*.

• Objects orbiting counterclockwise when seen from above the Earth’s North Pole are generally assigned inclinations between 0° and 90° (*prograde* or *direct*), those orbiting CW, between 90° and 180° (*retrograde*).
**Node**: where an orbit crosses the reference plane
There are two nodes, **ascending** and **descending**.
\( \Omega = \text{longitude of the ascending node} \)

**perihelion**: closest point to the Sun
**periapse**: closest point to central body
**aphelion** (“af-elion”): furthest point from Sun
**apoapse**: furthest point from central body
\( \omega = \text{argument of perihelion} \)
Orbit equation (polar coordinates)

\[ \frac{1}{r} = \frac{GM}{L^2} (1 + e \cos(\theta - \omega)) \]

We can replace \( L \) (the angular momentum, which is constant) by a different constant to make the equation cleaner. \( L \) and the new constant \( a \) are related by

\[ L^2 = GMa (1 - e^2) \]

This results (not coincidentally) in \( a \) being the semi-major axis (half the long axis) of the ellipse. If \( e=0 \) (the orbit is circular), \( a \) is the radius of the orbit.
The radial equation of motion

\[
\frac{1}{r} = \frac{GM}{L^2} \left(1 + e \cos(\theta - \varpi)\right)
\]

Using our substitution for L, the radial equation is thus reduced to its usual form, with \( f = \theta - \varpi \) (the true anomaly = the angle from periapse) used.

\[
r = \frac{a(1-e^2)}{1 + e \cos f}
\]

Note: \( \varpi \) is a constant called the \textit{longitude of perihelion} (“omega bar”, “pomega”, “curly pi”) and is related to \( \Omega \) and \( \omega \)

\[ \varpi \equiv \Omega + \omega \]
Energy

• The shape of the orbit can be expressed in terms of total energy \( E \) of the orbit, which is a conserved quantity

\[
E = \frac{1}{2} \mu v^2 - \frac{GM\mu}{r} = -\frac{GM\mu}{2a}
\]

• If \( E < 0 \), then the orbit is bound (0 < \( e < 1 \))
• if \( E > 0 \), then the orbit is hyperbolic (\( e > 1 \))
• if \( E=0 \), kinetic and potential energy are just balanced (\( e=1 \))
  – We can use this to compute \( v_e \), the escape velocity, the minimum velocity to escape from a particular distance from the central body (without further energy input)
  – the escape velocity turns out to be \( \sqrt{2} \) times the circular velocity
Kepler’s First Law

Bodies orbit the Sun on elliptical paths.
Kepler’s Second Law

- Conservation of angular momentum $L$ means that the area $A$ swept out by the radius vector during an interval of time $dt$ is constant since $L$ is constant:

$$L = \text{constant}$$
The orbital period

- The time period \( P \) required for one orbit can be derived from the angular momentum equation but we won’t bother here. This is Kepler’s third law.

\[
G(m_1 + m_2)P^2 = 4\pi^2 a^3
\]

- The mean motion \( n \) (mean angular frequency) of the orbit is just \( n = 2\pi/P \)

- This allows us to define the mean anomaly \( M = n(t-t_0) \) where \( t_0 \) is the time of perihelion passage. Note that the mean anomaly \( M \) is NOT equal to the true anomaly \( f \). Converting from one to the other is an involved process, though one that can be handled numerically with ease.
- We often also use the mean longitude \( \lambda = M + \omega \)

\[
P = a^{3/2}
\]

Note: if \( m_1 + m_2 = M_{\text{sun}} \), \( P \) is in years and \( a \) is in AU, then the simple equation above applies.
Kepler’s Third Law

tells us that the period of a body revolving about the Sun depends only on its major axis.
Some other useful quantities

\[ r = \frac{a(1-e^2)}{1+e \cos f} \]

From the radial equation we can immediately see that \( r \)
is a minimum at \( f = 0 \) and a max at \( f = \pi \). These points
are our *perihelion* \( q \) and *aphelion* (afelion) \( Q \).

\[ q = a(1-e) \quad \quad Q = a(1+e) \]

There are many more quantities than can be calculated
(eg. velocity at any point in the orbit) but we don’t have
time to go through them all here. Consult a celestial
mechanics textbook.
Orbital elements

• To describe a particle in orbit around the Sun requires 6 numbers: x, y, z position and x, y and z velocity

• We can describe the motion using other numbers. A commonly used set is called the standard *orbital elements*
  - a: semi-major axis
  - e: eccentricity
  - i: inclination
  - Ω: longitude of the ascending node
  - ω: argument of perihelion
  - M or f: mean or true anomaly
The three-body problem

• The two-body problem can be solved exactly analytically. The three or more body (N) problem cannot.

• This can be traced to an insufficient number of constants and “integrals”
  – An analytic solution requires more integrals and constraints than degrees of freedom of the system: 6N for N bodies
  – There are only 10 integrals:
    • energy
    • 3 components of the angular momentum
    • 6 constants given by the initial posn and vel of the CofM
  – Two other constraints:
    • longitude of the particle at \( t_0 \)
    • the direction of perihelion (Runge-Lenz vector)

• Going to 3 bodies exceeds the number of constants and integrals available, so no exact solution is possible (except in certain limiting cases)
The restricted three-body problem

• Consider a system of particles $m_1 >> m_2 >> m_3$
• To a first approximation, $m_2$ will orbit $m_1$ with little or no effect from $m_3$
• If we officially declare $m_3$ will have no effect on $m_1$ or $m_2$, we are within a restricted three-body problem.
  – *circular three body problem*: $m_2$ is on a circular orbit
  – *coplanar three body problem*: $m_3$ orbits $m_1$ in the same plane as $m_2$
  – *elliptic three body problem*: $m_2$ is on an elliptical orbit
The circular restricted three-body problem

- The fact that \( m_3 \) does not affect \( m_1 \) and \( m_2 \) breaks most of the previous integrals: they become only approximate in this case. Energy and angular momentum are no longer (strictly) conserved.

- One new integral appears: \textit{Jacobi’s constant}

\[
C_J = 2n(x\dot{y} - y\dot{x}) + \frac{2Gm_1}{|\vec{r} - \vec{r}_1|} + \frac{2Gm_2}{|\vec{r} - \vec{r}_2|} - \dot{x}^2 + \dot{y}^2 + \dot{z}^2
\]

- Here it is given in the non-rotating frame in more usual units. The textbook gives it in the rotating frame.

- \( n = 2\pi/P \) is the mean motion of the planet
Tisserand invariant

- To a first approximation, we can consider Jupiter, the most massive of the planets, to be in a circular orbit around the Sun. Thus the $C_J$ of any comet (or other small body) should remain constant even if the body’s orbit is perturbed by a close encounter with this giant planet.

- We can translate our equation for $C_J$ into one using orbital elements (if we determine these elements when the body is far from Jupiter). We can show that a quantity, called the Tisserand parameter, should remain approximately constant.

\[
C' = \frac{1}{\alpha} + 2\sqrt{\alpha(1-e^2)} \cos i
\]

where $\alpha = a/a_p$ where $ap$ = planet’s $a$, other values are for small body

- Thus a newly discovered comet (whose orbital elements do not closely match any other known comets), could be identified with a previously-seen comet if they have the same Tisserand parameter. In this case, a close approach with Jupiter would have changed the orbit, but $C'$ should remain (nearly) constant.

- $C'$ is also the basis for a classification scheme for small body orbits.
The Lagrange points

- There are five points in the CRTBP where a particle placed at rest (in the rotating frame) will remain indefinitely. These are the Lagrange points.
Tadpole orbits

- Objects not precisely at the triangular Lagrange points will “orbit” those points (as seen in the rotating frame). These are called *tadpole orbits* or *Trojan orbits*.
- The linear Lagrange points are unstable, and objects not precisely at them will drift away on time scales of tens-hundreds of orbits.
Horseshoe orbits

- Particles further from the triangular Lagrange points may find that the tail of their tadpole crosses the planet-Sun line and merges with the tadpole on the other side. These are called *horseshoe orbits*.
The Hill sphere

- The region around a planet where the planet’s gravitational influence dominates that of the Sun’s is approximately delimited by the **Hill sphere**.
- In the limit \( m_p << M_{\text{sun}} \), this is the same as the Roche lobe (not Roche limit):

\[
R_H = \left( \frac{m_p}{3(M_{\text{sun}} + m_p)} \right)^{1/3} a_p
\]

- Stable satellites must have orbits well within the Hill sphere. Orbits outside are in orbit around the Sun. Satellite orbits large enough to take them near the Hill sphere are generally unstable.
- Eg. For the Earth, \( R_H = 0.01 \text{ AU} \) or 1.5 million km. The Moon orbits stably at \( a = 384,000 \text{ km} \) or \( 0.25R_H \).
Perturbations

- In the Solar System, the orbits of planets, asteroids and comets are very close approximations to Keplerian ellipses most of the time (in the absence of close approaches to each other).

- However, the presence of other massive bodies does perturb their orbits from pure ellipses, resulting in a variety of different behaviours, even for the very massive planets, over the long-term.

- The orbits of the planets change over time(!)
The disturbing function

Consider masses $m_c, m_i$ and $m_j$, with $m_c \gg m_i$, and $m_c \gg m_j$. We can write the equations of motion of $m_i$ relative to $m_c$ as

\[
\ddot{\vec{r}}_i = -G(m_c + m_i) \frac{\vec{r}_i}{r_i^3} + Gm_j \left( \frac{\vec{r}_j - \vec{r}_i}{|\vec{r}_j - \vec{r}_i|^3} - \frac{\vec{r}_j}{r_j^3} \right)
\]

(note that by simply swapping subscripts we get the eqn for $m_j$)

The first term on the right is the two-body Keplerian term due to the central body. The second term is the effect of body $m_j$ on $m_i$. Note that the first term goes like $m_c$ while the second goes like the much smaller $m_j$. The addition of more bodies results in more terms on the right like the one for $m_j$. 


The disturbing function

- The (often small) perturbation due to another body (or many other bodies) is usually written as a potential. We can show that

\[
\ddot{r}_i = \nabla_i (U_i + R_i) = \left( \hat{i} \frac{\partial}{\partial x_i} + \hat{j} \frac{\partial}{\partial y_i} + \hat{k} \frac{\partial}{\partial z_i} \right) (U_i + R_i)
\]

where

\[
U_i = \frac{G(m_c + m_i)}{r_i}
\]

\[
R_i = \frac{Gm_j}{|\vec{r}_j - \vec{r}_i|} - Gm_j \frac{\vec{r}_i \cdot \vec{r}_j}{r_j^3}
\]

- \(R_i\) is the **disturbing function**, one of the most studied functions in celestial mechanics.
- The first term of \(R\) is called the **direct term** and reflects the direct pull of the other body.
- The second term is called the **indirect term**, reflecting the effect of the second body through its acceleration of the Sun.
- Note that the direct term gets small as \(r_i\) gets large, but the indirect term does not.
Precession

• The disturbing function is usually small compared to the Kepler term. As a result, the effect of the disturbing function is often small. For example, the motion may continue to be essentially a Keplerian ellipse, but with orbital elements that change slowly over time in response to the perturbations.

• One common result of the presence of a perturbing body is precession. It can be precession of the nodes (effectively, changing $\Omega$ with time) or of perihelion (effectively, changing $\omega$ or $\varpi$ over time).
Precession

- Precession of the line of nodes
- Precession of the line of apsides
- Animation of hypothetical Martian satellites
Precession in the inner Solar System
The disturbing function

• Though the effects of the disturbing function are usually small, they can provide a rich spectrum of behaviour.
• Note we cannot solve the motion due to the disturbing function analytically, which would be equivalent to solving the three-body problem itself.
• In general, we can approximate the behaviour by expanding the disturbing function as a series and then taking the lowest order terms as a simplification.
• Poincare in the late 19th century showed that the convergence of these series was often very poor, so that high-order terms could sometimes be quite large and the results of these expansion were sometimes only valid over a limited time span.
• As a result, we actually still don’t know how long our Solar System will remain stable(!)
Analytic vs. numerical

- Analytic approximations to the three-body problem, usually obtained through perturbation theory, can yield some insights (e.g. allow precession rates to be calculated).
- However the essential non-integrability of the multi-body problem means that analytic solutions can only take us so far.
- Numerical solutions to this problem are needed to study the Solar System in detail.
  - These methods have their own drawbacks but are simple in principle. Put your bodies down with the posns and vels you desire, calculate the forces between them, use these forces to calculate their accelerations over some small time step, reposition the bodies based on these accelerations, then repeat for as long as you like.
  - Done correctly, this method can give you information about the long-term behaviour of the system that analytic approaches can’t
Regularity and Chaos

• Where the perturbation series work well, the motion is called *regular* and the changes in the elements can usually be expressed as some simple functions of time (e.g. precession)

• Where the series do not work, the motion is often found to be *chaotic*. *Chaos* does not imply randomness, just high sensitivity to the initial conditions.
  – the butterfly effect
Chaos

• In regions of phase space where orbits are chaotic, two orbits with very similar initial conditions will diverge (in phase space or orbital element space) from each other exponentially, while the divergence is generally linear for regular motion.

• Generally, the easiest way to determine whether or not an orbit is chaotic is from numerical experiment: We place two particles near each other in the suspected chaotic region and watch how quickly they diverge.
Lyapunov timescale

- We can define the Lyapunov characteristic exponent $\gamma_c$ which is a measure of how fast the trajectories diverge. If $\gamma_c > 0$ then the system is chaotic ($< 0$ = regular).
- For chaotic motion, the Lyapunov timescale $1/\gamma_c$ is a measure of how long we can expect to reliably predict the motion of a particle in this region. Beyond a few Lyapunov times, our predictions are unreliable.

\[ d(t) \sim d(0)e^{\gamma_c t} \]

\[ \gamma_c \sim \frac{\ln(d(t)) - \ln(d(0))}{t} \]

Whether or not the motion is chaotic usually has to be determined by numerical experiment (eg numerical computation of the Lyap exponent from $d(t)$)

\[ d(0) = \text{initial distance between two simulated particles} \]
\[ d(t) = \text{distance at some future time } t \]
Determining of the Lyapunov time scale

\[ \frac{1}{\gamma_c} \sim \frac{t}{\ln(d(t)) - \ln(d(0))} \]

For regular motion, the plot will continue to increase roughly linearly.

For chaotic motion, the plot will eventually level off at the Lyapunov exponent, in this case about 350 years. As a result, we need to be careful predicting the motion of this asteroid on timescales >> 1/\(\gamma_c\).

Pitfalls:
• you usually have to go at least \(\sim 10\gamma_c\) to see the break clearly
• the particle can move between regions with different Lyapunov times, confusing the picture.
Resonances

• A particle in orbit around the Sun is an oscillator with a particular frequency, namely its orbital frequency.

• If another particle is also orbiting the Sun and assuming the two orbits are “far apart”, the second body perturbs the first particle, and does so with a particular frequency, namely its orbital frequency.

• Though planets are not harmonic oscillators, we do have a similar situation: an oscillator with a particular natural frequency is being driven at a different frequency.
Driven harmonic oscillator

- A driven harmonic oscillator of natural frequency $\omega_0$ driven at $\omega_d$ has a solution
  
  $$x = \frac{F_d}{m(\omega_0^2 - \omega_d^2)} \cos \omega_d t + C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

- The motion is mostly at $\omega_0$ but can become dominated by the first term if $\omega_0 \sim \omega_d$, even if $F_d$ is small. In the case $\omega_0 = \omega_d$, the solution becomes
  
  $$x = \frac{F_d}{2m\omega_0} t \cos \omega_0 t + C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

- Note the first term, which now grows with $t$. It becomes an extremely large and growing oscillation.
- In practice, particles in orbit can also show dramatic responses to even small perturbations, if they are applied near the orbital frequency (or a small integer ratio of the orbital frequency)
1:1 mean-motion resonance

- When a particle has nearly the same orbital frequency as another, they are said to be in one-to-one mean motion resonance.

- Some examples of these are objects at a planet’s Lagrange points, tadpole orbits and horseshoe orbits, discussed earlier.

Note in this case, the resonance actually has a stabilizing effect. This is because the equations of motion for orbiting particles are not the same as for the harmonic oscillator. Whether a given resonance is stabilizing or destabilizing has to be determined on a case by case basis.
Mean-motion resonances

- Mean-motion resonances also occur whenever \( P_1/P_2 = \text{ratio of integers} \) (e.g. 1:1, 2:3, 1:2, 1:3, 2:5,...) for two solar system bodies.
- In principle, the solar system is riddled with resonances, as any integer ratio (1001:361, 1544656:2049856, etc) constitutes a resonance.
- However the strength of the resonance typically goes down rapidly at higher orders. For example, the strength of a 2:1 resonance between a planet with \( e_1 \) and a particle with \( e_2 \) might go like \( e_1^2e_2 \) while the 5:7 resonance might go like \( e_1^5e_2^7 \) so in practice only the low-order resonances are effective (since usually \( e << 1 \)).
Laplace resonance

- This occurs when three bodies have orbits in simple integer ratios of each other.
- The most famous example is the moons Io, Europa and Ganymede of Jupiter, which are locked into a 1:2:4 orbital period ratio.
Where are they?

<table>
<thead>
<tr>
<th></th>
<th>$a$ ($10^6$ km)</th>
<th>$P$ (days)</th>
<th>$e$</th>
<th>$m_s$ ($10^{20}$ kg)</th>
<th>$R_s$ (km)</th>
<th>$\rho$ (Mg m$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Io</td>
<td>422</td>
<td>1.769</td>
<td>.041</td>
<td>893</td>
<td>1821</td>
<td>3.53</td>
</tr>
<tr>
<td>Europa</td>
<td>671</td>
<td>3.552</td>
<td>.010</td>
<td>480</td>
<td>1565</td>
<td>2.99</td>
</tr>
<tr>
<td>Ganymede</td>
<td>1070</td>
<td>7.154</td>
<td>.0015</td>
<td>1482</td>
<td>2634</td>
<td>1.94</td>
</tr>
<tr>
<td>Callisto</td>
<td>1883</td>
<td>16.69</td>
<td>.007</td>
<td>1076</td>
<td>2403</td>
<td>1.85</td>
</tr>
</tbody>
</table>
Laplace Resonance

- Periods of Io:Europa:Ganymede are in ratio 1:2:4
- This means that successive conjunctions occur at the same point on the orbit
- So the eccentricities get pumped up to much higher values than if the satellites were not in a resonance

![Diagram of the Laplace Resonance]

- High eccentricities mean higher tidal dissipation in the satellites and a tendency for the orbits to contract
- This tendency is counteracted by dissipation in Jupiter, which tends to cause the orbits to expand (like the Moon)
- The system is currently (roughly) in equilibrium
Kirkwood gaps

- A plot of the semi-major axis of asteroids in the main-belt, reveals many gaps at (destabilizing) mean-motion resonances with Jupiter.
- A few MMR actually show “accumulations” of asteroids (eg. Hilda’s at 2:3)
- Note: these gaps are not visible in a simple plot of the positions of the asteroids at a given time, due to the eccentricities of the asteroids.
Critical argument

• How can you determine if a particle is in resonance?
  – The periods do not need to be exactly the same, just close enough. But what’s close enough?
  – Numerical experiments are the easiest way to find out.
• If, say, $n_1/n_2$ (recall $n = 2\pi/P$) = 1/2, then $2\lambda_1 = \lambda_2 + C$ or $2\lambda_1 - \lambda_2 - C = 0$ in perfect resonance (The C allows that the perihelia of the two orbits may not be aligned)
• The quantity $2\lambda_1 - \lambda_2 - C = \phi$ is called the critical argument
• In general if $n_1/n_2 = p/(p+q)$, then the critical argument for the mean motion resonance is

$$\phi = (p + q)n_1 - pn_2 - C$$

• recall mean longitude $\lambda = M + \omega$
Critical argument

- If a particle is perfectly in resonance, a plot of $\phi$ will be a flat line.
- A particle trapped less deeply in resonance will have a critical argument that librates (oscillates).
- A non-resonant particle has a critical argument that circulates.

$$\phi = (p + q)n_1 - pn_2 - C$$
Secular perturbations

• The disturbing function can generally be expanded into a series of terms
  – some involve the mean longitudes (or anomalies) of the planets. These are called the *mean-motion terms*, and account for mean-motion resonances
  – the remaining terms do not involve the mean longitudes, and as a result, change much more slowly over time. These terms are called the *secular terms*, and represent the disturbing function as averaged over time.
Secular perturbations

• Generally produce precession in both $\varpi$ and $\Omega$ and associated oscillations with the same period in $e$ ($\varpi$) and $i$ ($\Omega$). There are generally several superimposed oscillations of different amplitudes.

• In computing the solution for a small body in our solar system, we see that each planet contributes 2 fundamental (the same for all bodies) frequencies to the final solution: one for $e-\varpi$ called $g_i$ and one for $i-\Omega$ called $f_i$ (sometimes $s_i$) where $i$ is a number 1-8 indicating the planet (Pluto is excluded due to its low mass).

• There is also one free precession frequency for each of $\varpi$ and $\Omega$ (for massless particles only), which depend on the properties of the particle orbit itself and in particular on its semi-major axis.

• If the free precession frequency equals one of the fundamental frequencies, a secular resonance occurs
Secular resonances

• the strong secular resonances are 1:1, sometime called the *linear secular resonances*

• if the free precession frequency of $\varpi$ of a massless particle equals $g_i$, then the $\nu_i$ resonance is active
  – eg. $\nu_5$ (“nu-five”) resonance means the small body’s free precession of $\varpi$ is at the same rate as the $g_5$ frequency (effectively but not exactly Jupiter’s own $\varpi$ precession rate)

• if the free precession of $\Omega$ equals $f_i$, then the $\nu_{1i}$ resonance is said to be active
  – eg. $\nu_{16}$ (“nu-sixteen”) resonance means the small body’s free precession of $\Omega$ is at the same rate as the $f_6$ frequency (effectively but not exactly Saturn’s own $\Omega$ precession rate)

• Secular resonances, like MM resonances, can result in disproportionately large changes in the orbital elements.
Fig. 7.19. The location of the important $\nu_5$, $\nu_6$, and $\nu_{16}$ linear secular resonances (calculated using $e_{\text{proper}} = 0.1$) and the numbered asteroids' $I_{\text{proper}}$ as a function of proper semi-major axis.
FIGURE 2.9 Schematic illustration of the tidal forces of a moon on a deformable planet. (a) Gravitational force of the moon on different parts of the planet. (b) Differential force of the moon’s gravity relative to the force on the planet’s center of mass. (c) Response of the planet’s figure to the moon’s tidal pull.

\[
\frac{2GmR}{x^3} \approx F_1 - F_{cm} = F_1 - F_{cm} = \frac{2GmR}{x^3}
\]
The Roche limit

- Tidal forces increase strongly with relative distance \( (1/r^3) \)
- The Roche limit is the critical distance between a small body and a planet where the tidal forces will tear the body apart \textit{if it is held together only by its own gravity.}

\[
r_R = 2.44 R_p \left( \frac{\rho_p}{\rho_s} \right)^{1/3}
\]

- \( R_p \) is the planet’s radius, and \( \rho_p \) and \( \rho_s \) are the planet’s and small body’s densities respectively
- This does not take into account the small body’s internal strength and is strictly only applicable to liquid or “rubble-pile” bodies (upper limit).
Radiation pressure

- A particle illuminated by the Sun on one side receives a momentum flux. This results in a net force radially outward on the particle, called radiation pressure.
- There is also an effect due to the solar wind (called corpuscular drag in the text), but it is generally much smaller except for very small grains.
- The ratio of the gravitational force from the Sun and the radiation pressure is constant (both drop as $1/r^2$) and is called $\beta$.
- A particle with $\beta>1$ is called a beta meteoroid and will be ejected from the Solar system by radiation pressure (typically sub-micron sized particles).

FIGURE 2.14 Ratio of corpuscular drag (caused by the solar wind) to that resulting from solar radiation, $\beta_{CP}$, is plotted against radius for grains composed of obsidian. (Adapted from Burns et al. 1979)
A particle warmed by the Sun and small enough to come to a constant temperature throughout will radiate isotropically in its own rest frame.

This radiation has a small Doppler shift in the Sun’s reference frame owing to the particle’s orbital motion.

As a result, energy and momentum are not radiated isotropically in the rest frame of the Sun: more is radiated forward and less backwards.

As a result, the particle feels a net retarding force, known as Poynting-Robertson drag.

**Poynting-Robertson drag**

![Diagram of Poynting-Robertson drag](image)

FIGURE 2.13 A particle in heliocentric orbit, that reradiates the solar energy flux isotropically in its own frame of reference, preferentially emits more momentum, $p$, in the forward direction as seen in the solar frame, because the frequencies and momenta of the photons emitted in the forward direction are increased by the particle’s motion. (Adapted from Burns et al. 1979)
Poynting-Robertson drag

- Poynting-Robertson drag can also be characterized by $\beta$.

$$\beta = \frac{\text{radiation pressure}}{\text{Solar gravity}}$$

$$\beta \propto \frac{R^2 / r^2}{m / r^2} \text{ where } R = \text{particle radius}, \ m = \text{particle mass}, \ r = \text{distance from Sun}$$

$$\beta \propto \frac{R^2 / r^2}{(\rho R^3) / r^2} \propto \frac{1}{R}$$

- So for roughly spherical particles, radiation pressure becomes more important as particle size decreases.

- Radiation pressure has significant effects on cm-sized (and smaller) particles, which spiral inwards.

- As a result, small grains in the SS cannot be primordial. If they exist in our SS (which they do, see comets) they must be continuously regenerated.
Yarkovsky effect

• A larger body in orbit around the Sun will not come to an equal temperature throughout: it will be warmer on one side and cooler on the other.

• If it is rotating, the object will reradiate thermally more in one direction than the other, producing a net force on the body.

• If the rotation is retrograde, the Y effect will produce a drag similar to PR drag.

• If the rotation is prograde, the force is in the opposite direction, and the body will spiral outwards.

• The Y effect can also affect the spin state of the body, slowing or speeding it up.

• The size of the Y effect depends on many parameters (albedo, rotation, shape, etc) but is important for objects in the m to km range.